1.4 - SMALL SIGNAL MODEL OF THE BJT

INTRODUCTION

Objective
The objective of this presentation is:
1.) Concept of the small signal model
2.) The small signal model for the BJT

Outline
• Transconductance small signal model
• Input resistance, output resistance of the common emitter model
• Extensions of the small signal BJT model
• BJT frequency response
## TRANSCONDUCTANCE SMALL SIGNAL MODEL

### Categorization of Electrical Models

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<th>Linearity</th>
<th>Time Dependence</th>
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<td>Linear</td>
<td>Time Independent: Small-signal, midband $R_{in}$, $A_v$, $R_{out}$ (.TF)</td>
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<td>Time Dependent: Small-signal frequency response - poles and zeros (.AC)</td>
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<td>Nonlinear</td>
<td>Time Independent: DC operating point $i_D = f(v_D,v_G,v_S,v_B)$ (.OP)</td>
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<td>Time Dependent: Large-signal transient response - Slew rate (.TRAN)</td>
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Based on the simulation capabilities of SPICE.
What is a Small Signal Model?

- A small signal model is a linear model which is independent of amplitude. It may or may not have time dependence (i.e. capacitors).
- The small signal model for a nonlinear component such as a BJT is a linear model about some nominal operating point. The deviations from the operating point are small enough that it approximates the nonlinear component over a limited range of amplitudes.

Illustration of the $pn$ diode:
**BJT, Common-Emitter, Forward-Active Region**

Effect of a small-signal input voltage applied to a BJT.

\[ i_C = I_C + i_c \]

\[ i_B = I_B + i_b \]

\[ v_i \Rightarrow i_b \Rightarrow i_c \]

\[ n_p(0) = n_{po} \exp \left( \frac{V_{BE} + V_{be}}{V_t} \right) \]

\[ n_p(0) = n_{po} \exp \left( \frac{V_{BE}}{V_t} \right) \]

\[ v_i \gg i_b \gg i_c \]

\[ V_{CC} \]

\[ V_{BE} \]

\[ V_{BE} + V_{be} \]

\[ V_{CC} \]

\[ V_{BE} \]

\[ I_C + i_c \]

\[ I_C \]

\[ \Delta Q_h \]

\[ \Delta Q_e \]

\[ W_B \]

\[ x \]

**Fig. 1.4-2**
**Transconductance of the Small Signal BJT Model**

The small signal transconductance is defined as

\[
g_m = \frac{di_C}{dv_{BE}} \bigg|_{Q} = \frac{i_c}{\Delta v_{BE}} = \frac{i_c}{v_{be}} = \frac{i_c}{v_i} \quad \Rightarrow \quad i_c = g_m v_i
\]

The large signal model for \( i_C \) is

\[
i_C = I_S \exp \left( \frac{v_{BE}}{V_t} \right) \quad \Rightarrow \quad g_m = \left( \frac{d}{dv_{BE}} I_S \exp \frac{v_{BE}}{V_t} \right) \bigg|_{Q} = I_S \frac{V_{BE}}{V_t} \exp \frac{V_{BE}}{V_t} = \frac{I_C}{V_t}
\]

\[\therefore \quad g_m = \frac{I_C}{V_t}\]

Another way to develop the small signal transconductance

\[
i_C = I_S \exp \left( \frac{V_{BE}+v_i}{V_t} \right) = I_S \exp \left( \frac{V_{BE}}{V_t} \right) \exp \left( \frac{v_i}{V_t} \right) = I_C \exp \left( \frac{v_i}{V_t} \right) = I_C \left[ 1 + \frac{v_i}{V_t} + \frac{1}{2} \left( \frac{v_i}{V_t} \right)^2 + \frac{1}{6} \left( \frac{v_i}{V_t} \right)^3 + \cdots \right]
\]

But

\[
i_C = I_C + i_c \]

\[\therefore \quad i_c \approx I_C \frac{v_i}{V_t} + \frac{I_C}{2} \left( \frac{v_i}{V_t} \right)^2 + \frac{I_C}{6} \left( \frac{v_i}{V_t} \right)^3 + \cdots \approx \frac{I_C}{V_t} v_i = g_m v_i\]
INPUT AND OUTPUT RESISTANCE SMALL SIGNAL MODEL

Input Resistance of the Small Signal BJT Model

In the forward-active region, we can write that

\[ i_B = \frac{i_C}{\beta_F} \]

Small changes in \(i_B\) and \(i_C\) can be related as

\[ \Delta i_B = \frac{d}{di_C} \left( \frac{i_C}{\beta_F} \right) \Delta i_C \]

The small signal current gain, \(\beta_o\), can be written as

\[ \beta_o = \frac{\Delta i_C}{\Delta i_B} = \frac{1}{\frac{d}{di_C} \left( \frac{i_C}{\beta_F} \right)} = \frac{i_C}{i_b} \]

Therefore, we define the small signal input resistance as

\[ r_\pi \equiv \frac{v_i}{i_b} = \frac{\beta_o v_i}{i_c} = \frac{\beta_o}{g_m} \]

\[ r_\pi = \frac{\beta_o}{g_m} \]
**Output Resistance of the Small Signal BJT Model**

In the forward-active region, we can write that the small signal output conductance, \( g_o \) \((r_o = 1/g_o)\) is

\[
g_o \equiv \frac{di_C}{dv_{CE}} \bigg|_Q = \frac{\Delta i_C}{\Delta v_{CE}} = \frac{i_c}{v_{ce}} \quad \Rightarrow \quad i_c = g_o v_{ce}
\]

The large signal model for \( i_C \), including the influence of \( v_{CE} \), is

\[
i_C = I_S \left( 1 + \frac{v_{CE}}{V_A} \right) \exp \frac{v_{BE}}{V_t}
\]

\[
g_o \equiv \frac{di_C}{dv_{CE}} \bigg|_Q = I_S \left( \frac{1}{V_A} \right) \exp \frac{V_{BE}}{V_t} \approx \frac{I_C}{V_A}
\]

\[
\therefore \quad r_o = \frac{V_A}{I_C}
\]
Simple Small Signal BJT Model

Implementing the above relationships, $i_c = g_m v_i$, $i_c = g_o v_{ce}$, and $v_i = r_\pi i_b$, into a schematic model gives,

![Schematic diagram of a BJT model.]

Note that the small signal model is the same for either a $n$pn or a $p$np BJT.

Example:

Find the small signal input resistance, $R_{in}$, the output resistance, $R_{out}$, and the voltage gain of the common emitter BJT if the BJT is unloaded ($R_L = \infty$), $v_{out}/v_{in}$, the dc collector current is 1mA, the Early voltage is 100V, and $\beta_0$ at room temperature.

$$g_m = \frac{I_C}{V_t} = \frac{1\text{mA}}{26\text{mV}} = \frac{1}{26} \text{ mhos or Siemans}$$

$$R_{in} = r_\pi = \frac{\beta_0}{g_m} = 100 \cdot \frac{26}{1} = 2.6k\Omega$$

$$R_{out} = r_o = \frac{V_A}{I_C} = \frac{100V}{1\text{mA}} = 100k\Omega$$

$$\frac{v_{out}}{v_{in}} = -g_m r_o = -26\text{mS} \cdot 100\text{k}\Omega = -2600\text{V/V}$$
EXTENSIONS OF THE SMALL SIGNAL BJT MODEL

Collector-Base Resistance of the Small Signal BJT Model

Recall the influence of $V$ on the base width:

We noted that an increase in $v_{CE}$ causes an increase in the depletion width and a decrease in the total minority-carrier charge stored in the base and therefore a decrease in the base recombination current, $i_{B1}$.

This influence is modeled by a collector-base resistor, $r_\mu$, defined as

$$r_\mu = \frac{\Delta v_{CE}}{\Delta i_{B1}} = \frac{\Delta v_{CE}}{\Delta i_C} \frac{\Delta i_C}{\Delta i_{B1}} = r_o \frac{\Delta i_C}{\Delta i_{B1}} = \beta_o r_o \quad \text{(lower limit if base current is all recombination current)}$$

In general, $r_\mu \geq 10 \beta_o r_o$ for the $npn$ BJT and about 2-5 $\beta_o r_o$ for the lateral $pnp$ BJT.
**Base-Charging Capacitance of the Small Signal BJT Model**

Consider changes in base-carrier concentrations once again.

The $\Delta V_{BE}$ change causes a change in the minority carriers, $\Delta Q_e = q_e$, which must be equal to the change in majority carriers, $\Delta Q_h = q_h$. This charge can be related to the voltage across the base, $v_i$, as

$$q_h = C_b v_i$$

where $C_b$ is the base-charging capacitor and is given as

$$C_b = \frac{q_h}{v_i} = \frac{\tau_F i_c}{v_i} = \frac{\tau_F g_m}{V_t} = \frac{I_C}{W_B}$$

The base transit time $\tau_F$ is defined as $\frac{W_B^2}{2D_n}$
Parasitic Elements of the BJT Small Signal Model

Typical cross-section of the npn BJT:

\[ C_{je} = \text{base-emitter depletion capacitance (forward biased)} \]

\[ C_{\mu} = \frac{C_{\mu 0}}{1 - \frac{v_C B}{\psi_0}^m} = \text{collector-base depletion capacitance (reverse biased)} \]

Resistances are all bulk ohmic resistances. Of importance are \( r_b, r_c, \) and \( r_{ex}. \) Also, \( r_b \) is a function of \( I_C. \)
The capacitance, $C_\pi$, consists of the sum of $C_{je}$ and $C_b$.

$$C_\pi = C_{je} + C_b$$
**Example**

Derive the complete small signal equivalent circuit for a BJT at $I_C = 1\text{mA}$, $V_{CB} = 3\text{V}$, and $V_{CS} = 5\text{V}$. The device parameters are $C_{je0} = 10\text{fF}$, $n_e = 0.5$, $\psi_{0e} = 0.9\text{V}$, $C_{\mu 0} = 10\text{fF}$, $n_c = 0.3$, $\psi_{0c} = 0.5\text{V}$, $C_{cs0} = 20\text{fF}$, $n_s = 0.3$, $\psi_{0s} = 0.65\text{V}$, $\beta_o = 100$, $\tau_F = 10\text{ps}$, $V_A = 20\text{V}$, $r_b = 300\Omega$, $r_c = 50\Omega$, $r_{ex} = 5\Omega$, and $r_{\mu} = 10\beta_or_o$.

**Solution**

Because $C_{je}$ is difficult to determine and usually an insignificant part of $C_\pi$, let us approximate it as $2C_{je0}$.

$$C_{je} = 20\text{fF}$$

$$C_\mu = \frac{C_{\mu 0}}{1 + \left(\frac{V_{CB}}{\psi_{0c}}\right)n_e} = \frac{10\text{fF}}{1 + \left(\frac{3}{0.5}\right)0.3} = 5.6\text{fF} \quad \text{and} \quad C_{cs} = \frac{C_{cs0}}{1 + \left(\frac{V_{CS}}{\psi_{0s}}\right)n_s} = \frac{20\text{fF}}{1 + \left(\frac{5}{0.65}\right)0.3} = 10.5\text{fF}$$

$$g_m = \frac{I_C}{V_t} = \frac{1\text{mA}}{26\text{mV}} = 38\text{mA/V} \quad C_b = \tau_F g_m = (10\text{ps})(38\text{mA/V}) = 0.38\text{pF}$$

$$C_\pi = C_b + C_{je} = 0.38\text{pF} + 0.02\text{pF} = 0.4\text{pF}$$

$$r_\pi = \frac{\beta_o}{g_m} = 100 \cdot 26\Omega = 2.6k\Omega \quad r_o = \frac{V_A}{I_C} = \frac{20\text{V}}{1\text{mA}} = 20k\Omega \quad \text{and} \quad r_{\mu} = 10\beta_or_o = 10 \cdot 100 \cdot 20k\Omega = 20M\Omega$$
**FREQUENCY RESPONSE OF THE BJT**

**Transition Frequency, \( f_T \)**

\( f_T \) is the frequency where the magnitude of the short-circuit, common-emitter current equal unity.

Circuit and model:

Assume that \( r_c \approx 0 \). As a result, \( r_o \) and \( C_{cs} \) have no effect.

\[
V_1 \approx \frac{r_\pi}{1 + r_\pi (C_\pi + C_b)s} I_i \quad \text{and} \quad I_o \approx g_m V_1 \quad \Rightarrow \quad \frac{I_o(j\omega)}{I_i(j\omega)} = \frac{g_m r_\pi}{1 + g_m r_\pi g_m} \frac{(C_\pi + C_b)s}{g_m} = \frac{\beta_o}{1 + \beta_o \frac{(C_\pi + C_b)s}{g_m}}
\]

Now,

\[
\beta(j\omega) = \frac{I_o(j\omega)}{I_i(j\omega)} = \frac{\beta_o}{1 + \beta_o \frac{(C_\pi + C_b)s}{g_m}}
\]

At high frequencies,

\[
\beta(j\omega) = \frac{g_m}{j\omega (C_\pi + C_b)} \quad \Rightarrow \quad \text{When } |\beta(j\omega)| = 1 \text{ then } \omega_T = \frac{g_m}{C_\pi + C_b} \quad \text{or} \quad f_T = \frac{1}{2\pi} \frac{g_m}{C_\pi + C_b}
\]
Illustration of the BJT Transition Frequency

$\beta$ as a function of frequency:

Note that the product of the magnitude and frequency at any point on the -6dB/octave curve is equal to $\omega_T$.

For example,

$$0.1 \omega_T \times 10 = \omega_T$$

In measuring $\omega_T$, the value of $|\beta(j\omega)|$ is measured at some frequency less than $\omega_T$ (say $\omega_x$) and $\omega_T$ is calculated by taking the product of $|\beta(j\omega_x)|$ and $\omega_x$ to get $\omega_T$. 
**Current Dependence of $f_T$**

Note that $\tau_T = \frac{1}{\omega_T} = \frac{C \pi}{g_m} + \frac{C \mu}{g_m} = \frac{C_b}{g_m} + \frac{C_{je} \mu}{g_m} + \frac{C_{\mu}}{g_m} = \tau_F + \frac{C_{je} \mu}{g_m} + \frac{C_{\mu}}{g_m}$

At low currents, the $C_{je}$ and $C_{\mu}$ terms dominate causing $\tau_T$ to rise and $\omega_T$ to fall.

At high currents, $\tau_T$ approaches $\tau_F$ which is the maximum value of $\omega_T$.

For further increases in collector current, $\omega_T$ decreases because of high-level injection effects and the Kirk effect.

Typical frequency dependence of $f_T$:

![Graph showing frequency dependence of $f_T$]